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A flow reversal due to edge effects

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An experiment is described which indicates a flow reversal due to fluid elasticity in the case of a rotating-plate experiment. An attempt to predict this reversal analytically on the 'infinite' plate assumption fails and recourse is made to a numerical procedure to solve the 'finite' plate problem. It is shown that the reversal is due to edge effects.

1. Introduction

In recent years there has been a growing interest in the flow characteristics caused by a solid of revolution rotating in an elastico-viscous liquid. Particular attention has been paid to the case of a sphere rotating in an infinite expanse of liquid (Thomas & Walters 1964; Walters & Savins 1965; Giesekus 1965) and to the rotating cone-stationary plate arrangement (Giesekus 1967; Walters & Waters 1968, p. 211). It has been deduced that an investigation of the secondary flow reversal caused by the elasticity in the liquids can lead to a determination of certain elastico-viscous parameters.

In the present paper, we investigate the flow caused by the slow rotation of a *finite* circular disk rotating about a vertical axis of symmetry in an elastico-viscous liquid, the liquid being otherwise confined by convenient bath surfaces. Such an investigation is of special interest since the theory based on the rotation of an *infinite* disk predicts no flow reversal of the type predicted and observed in other geometries (see §3), whereas an experimental investigation clearly indicates a reversal of flow throughout the liquid (see §2). This paper is concerned with an understanding of this unexpected flow reversal. Using a numerical procedure in §4, it is shown that the reversal is due to 'edge effects', in the sense that such a flow reversal takes place because of the 'finiteness' of the rotating plate. As a general conclusion, it is pointed out that edge effects can affect the flow characteristics throughout a flow field and not just in regions 'near the edges'.

2. Experiment

(a) *Description of the apparatus*

The experimental arrangement consisted essentially of a circular aluminium disk of radius 4·75 cm and thickness 0·2 cm rotating slowly about an axis through its centre perpendicular to the plane of the disk. The disk was rotated by means of a rod attached to its centre and connected at the other end to a motor-driven dynamometer. By means of an attachment on the dynamometer the disk could be rotated at various speeds.

The disk was submerged in the test solution, which was contained in a perspex tank. It was rotated near the base of the tank and the streamlines formed in the expanse of fluid between it and the bottom of the tank were studied. The streamlines were rendered visible by use of the 'water blue' dye.

(b) *Experimental observations*

The first experiment was performed using a purely viscous liquid, namely glycerol, with a viscosity of about 15 poises. The streamlines were observed at various rotational speeds and were seen to be directed outwards at the rotating plate, downwards to the bottom of the tank and then up towards the disk. The patterns formed can be seen in figure 1, plate 1. The experiment was repeated for various values of the gap width between the rotating plate and the tank base with the same results.

A 3% aqueous solution of polyacrylamide was used in the second experiment. The limiting viscosity of the solution at small rates of shear was approximately 50 poises and the plate was rotated at 6 rev/min. A reversal of the situation in the viscous case was found to occur; the streamlines being directed inwards at the rotating disk and radially outwards at the base of the tank. Figure 2, plate 1, illustrates the general shape of the streamlines. The dye was initially situated at the centre of the rotating disk and the direction of the secondary flow is clearly discernible, as is the appearance of a nodal position at a point approximately under the edge of the plate. It is interesting to note that the reversal in flow direction affects the whole of the flow field between the disk and the bottom of the tank.

3. Theory

(a) *Choice of equations of state*

In the solution of elastico-viscous flow problems, it is necessary to characterize the fluids by means of suitable equations of state. In general terms, it is true to say that most of the flow problems to be found in the literature have involved either the use of Oldroyd-type equations of state (see, for example, Oldroyd 1958) or the 'order' equations of Coleman & Noll (1961). The problem under consideration in the present paper involves a slow steady flow and under such conditions it can be shown (Walters 1969, to be published) that Oldroyd-type equations of state and the order equations are equivalent. We shall therefore use the (third-) order equations of Coleman & Noll, since in these the stress is given explicitly in terms of the kinematic variables. The relevant equations of state can be written in the form

$$p_{ik} = -pg_{ik} + p'_{ik}, \tag{1}$$

$$p'_{ik} = 2\alpha_1 e^{(1)}_{ik} + 2\alpha_2 e^{(2)}_{ik} + 4\alpha_3 e^{(1)j}_{i} e^{(1)}_{jk} + 2\beta_2 e^{(3)}_{ik} + 8\beta_1 e^{(1)l}_{j} e^{(1)j}_{l} e^{(1)}_{ik} + 4\alpha_5 (e^{(1)j}_{i} e^{(2)}_{jk} + e^{(2)j}_{i} e^{(1)}_{jk}), \tag{2}$$

where p_{ik} is the stress tensor, g_{ik} the metric tensor of a fixed co-ordinate system x^i, $e_{ik}^{(1)}$ the rate-of-strain tensor, $e_{ik}^{(n)} = \mathfrak{d}^{n-1}e_{ik}^{(1)}/\mathfrak{d}t^{n-1}$ is called the *nth* rate-of-strain tensor, $\mathfrak{d}/\mathfrak{d}t$ being the convected time derivative introduced by Oldroyd (1950). $e_{ik}^{(n)}$ is related to the nth Rivlin–Ericksen (1955) tensor $A_{ik}^{(n)}$ by the relation

$$A_{ik}^{(n)} = 2e_{ik}^{(n)}. \tag{3}$$

$\alpha_1, \alpha_2, \alpha_3, \alpha_5, \beta_1$ and β_2 are material constants and (confining attention to incompressible materials) p is an arbitrary isotropic pressure. In (1) and (2), covariant suffixes are written below, contravariant suffixes above, and the usual summation convention for repeated suffixes is implied.

For completeness, we note that if we had used Oldroyd-type equations of the form

$$\begin{aligned} p'_{ik} + \lambda_1 \mathfrak{d}/\mathfrak{d}t\, p'_{ik} + \mu_0 p'^{j}_{j} e_{ik}^{(1)} - \mu_1 (p'^{j}_{i} e_{jk}^{(1)} + p'^{j}_{k} e_{ji}^{(1)}) \\ = 2\eta_0 [e_{ik}^{(1)} + \lambda_2 e_{ik}^{(2)} - 2\mu_2 e_i^{(1)j} e_{jk}^{(1)}], \end{aligned} \tag{4}$$

where η_0, λ_1, λ_2, μ_0, μ_1 and μ_2 are material constants, the relevant equations would be the same as those given below with the following relation between the constants:

$$\left.\begin{aligned} \alpha_1 &= \eta_0, \\ \alpha_2 &= -\eta_0(\lambda_1 - \lambda_2), \\ \alpha_3 &= \eta_0(\mu_1 - \mu_2), \\ \beta_2 &= \eta_0 \lambda_1 (\lambda_1 - \lambda_2), \\ \beta_1 &= \tfrac{1}{2}\eta_0[\mu_0\{(\lambda_1 - \lambda_2) - (\mu_1 - \mu_2)\} + \mu_1(\mu_1 - \mu_2) + \lambda_1(\mu_1 - \mu_2)], \\ \alpha_5 &= \tfrac{1}{2}\eta_0[\mu_1\lambda_2 - 3\mu_1\lambda_1 + 2\lambda_1\mu_2]. \end{aligned}\right\} \tag{5}$$

(*b*) *Basic equations*

In this section, we consider the basic equations for the slow steady flow of an elastico-viscous liquid caused by the rotation of a general solid of revolution about a vertical axis. All physical quantities will be referred to cylindrical polar coordinates (r, θ, z), the z-axis coinciding with the axis of rotation. U, V and W will denote the physical components of the velocity vector in the r, θ, z directions, respectively.

We introduce the dimensionless variables† (cf. Thomas & Walters 1964; Walters & Waters 1968)

$$\left.\begin{aligned} U = \frac{\nu u}{h}, \quad V = \Omega h v, \quad W = \frac{\nu w}{h}, \\ p = \rho g z + \frac{\rho \nu^2}{h^2} p^*, \quad r = h r_1, \quad z = h z_1, \end{aligned}\right\} \tag{6}$$

† Brackets placed round suffixes are used to denote the physical components of tensors.

$$p'_{(ik)} = \frac{\alpha_1 \nu}{h^2} \begin{bmatrix} p''_{(r_1 r_1)} & \left(\frac{\Omega h^2}{\nu}\right) p''_{(r_1 \theta)} & p''_{(r_1 z_1)} \\ \left(\frac{\Omega h^2}{\nu}\right) p''_{(r_1 \theta)} & p''_{(\theta\theta)} & \left(\frac{\Omega h^2}{\nu}\right) p''_{(\theta z_1)} \\ p''_{(r_1 z_1)} & \left(\frac{\Omega h^2}{\nu}\right) p''_{(\theta z_1)} & p''_{(z_1 z_1)} \end{bmatrix}, \tag{7}$$

where g is the acceleration due to gravity, Ω the angular velocity of the solid of revolution, ρ the density of the fluid, h a typical length and $\nu = \alpha_1/\rho$. The equations of motion and continuity (for axial symmetry) can now be written in the form

$$u\frac{\partial u}{\partial r_1} + w\frac{\partial u}{\partial z_1} - \frac{Lv^2}{r_1} = -\frac{\partial p^*}{\partial r_1} + \frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1 p''_{(r_1 r_1)}) + \frac{\partial}{\partial z_1} p''_{(r_1 z_1)} - \frac{p''_{(\theta\theta)}}{r_1}, \tag{8}$$

$$\frac{u}{r_1}\frac{\partial}{\partial r_1}(r_1 v) + w\frac{\partial v}{\partial z_1} = \frac{1}{r_1^2}\frac{\partial}{\partial r_1}(r_1^2 p''_{(r_1 \theta)}) + \frac{\partial p''_{(\theta z_1)}}{\partial z_1}, \tag{9}$$

$$u\frac{\partial w}{\partial r_1} + w\frac{\partial w}{\partial z_1} = -\frac{\partial p^*}{\partial z_1} + \frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1 p''_{(r_1 z_1)}) + \frac{\partial p''_{(z_1 z_1)}}{\partial z_1}, \tag{10}$$

$$\frac{\partial}{\partial r_1}(r_1 u) + \frac{\partial}{\partial z_1}(r_1 w) = 0, \tag{11}$$

where
$$L = \left(\frac{\Omega h^2}{\nu}\right)^2, \tag{12}$$

i.e. the square of a Reynolds number.

In the following we shall obtain a solution in ascending powers of L, noting in the first place that, when terms of order L are neglected, a simple solution to the problem exists of the form

$$u = w = 0, \quad v = v_0. \tag{13}$$

v_0 satisfies the equation
$$\nabla_1^2 v_0 - \frac{v_0}{r_1^2} = 0, \tag{14}$$

where
$$\nabla_1^2 \equiv \frac{\partial}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial}{\partial r_1} + \frac{\partial^2}{\partial z_1^2}. \tag{15}$$

We now assume that the velocity components and the pressure can be expanded in ascending powers of L as follows:

$$\left.\begin{aligned} U &= \frac{\nu}{h}[Lu_1 + L^2 u_2 + \ldots], \\ V &= \Omega h[v_0 + Lv_1 + \ldots], \\ W &= \frac{\nu}{h}[Lw_1 + L^2 w_2 + \ldots], \\ p - \rho g z &= \frac{\rho \nu^2}{h^2}[Lp_1^* + L^2 p_2^* + \ldots]. \end{aligned}\right\} \tag{16}$$

In this section, we work to first order in L. Second-order terms will be considered in §3(*c*). The first-order stress components corresponding to (2) and (16) can be shown to be

$$\left.\begin{aligned}
p''_{(r_1r_1)} &= L\left[\frac{\partial u_1}{\partial r_1}+(2\alpha_2'+\alpha_3')r_1^2\left\{\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)\right\}^2\right],\\
p''_{(\theta\theta)} &= L\left[\frac{u_1}{r_1}+\alpha_3'\left\{r_1^2\left(\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)\right)^2+\left(\frac{\partial v_0}{\partial z_1}\right)^2\right\}\right],\\
p''_{(z_1z_1)} &= L\left[\frac{\partial w_1}{\partial z_1}+(2\alpha_2'+\alpha_3')\left(\frac{\partial v_0}{\partial z_1}\right)^2\right],\\
p''_{(\theta z_1)} &= \frac{\partial v_0}{\partial z_1}+L\left[\frac{\partial v_1}{\partial z_1}+\alpha_2'\left(\frac{u_1}{r_1}\frac{\partial}{\partial r_1}+\frac{w_1}{r_1}\frac{\partial}{\partial z_1}\right)r_1\frac{\partial v_0}{\partial z_1}+2(\alpha_2'+\alpha_3')\frac{u_1}{r_1}\frac{\partial v_0}{\partial z_1}\right.\\
&\quad+2\alpha_3' r_1\frac{\partial w_1}{\partial r_1}\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)+(\alpha_2'+\alpha_3')\left\{\frac{\partial w_1}{\partial z_1}\frac{\partial v_0}{\partial z_1}+r_1\frac{\partial u_1}{\partial z_1}\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)\right\}\\
&\quad\left.+2(\alpha_5'+\beta_1')\frac{\partial v_0}{\partial z_1}\left(r_1^2\left\{\frac{\partial}{\partial r_1}\left(\frac{v_0}{r}\right)\right\}^2+\left\{\frac{\partial v_0}{\partial z}\right\}^2\right)\right],\\
p''_{(r_1\theta)} &= r_1\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)+L\left[r_1\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)+\frac{\alpha_2'}{r_1}\left(u_1\frac{\partial}{\partial r_1}+w_1\frac{\partial}{\partial z_1}\right)\left[r_1^2\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)\right]\right.\\
&\quad+(\alpha_2'+2\alpha_3')\left(\frac{\partial u_1}{\partial r_1}+\frac{u_1}{r_1}\right)r_1\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)+(\alpha_2'+\alpha_3')\frac{\partial w_1}{\partial r_1}\frac{\partial v_0}{\partial z_1}+\alpha_3'\frac{\partial v_0}{\partial z_1}\frac{\partial u_1}{\partial z_1}\\
&\quad\left.+2(\alpha_5'+\beta_1')\left\{r_1^2\left(\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)\right)^2+\left(\frac{\partial v_0}{\partial z_1}\right)^2\right\}r_1\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)\right],\\
p''_{(r_1z_1)} &= L\left[\frac{\partial w_1}{\partial r_1}+\frac{\partial u_1}{\partial z_1}+(\alpha_2'+\alpha_3')r_1\frac{\partial v_0}{\partial z_1}\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)\right],
\end{aligned}\right\}\tag{17}$$

where $$\alpha_2' = \frac{\alpha_2}{\rho h^2},\quad \alpha_3' = \frac{\alpha_3}{\rho h^2},\quad \alpha_5' = \frac{\alpha_5\alpha_1}{\rho^2h^4},\quad \beta_1' = \frac{\beta_1\alpha_1}{\rho^2h^4}.$$

From the equation of continuity (11), we can define a stream function, χ, such that

$$u = -\frac{1}{r_1}\frac{\partial\chi}{\partial z_1},\quad w = \frac{1}{r_1}\frac{\partial\chi}{\partial r_1},\tag{18}$$

and we write $\chi = L\chi_1+L^2\chi_2+\dots$.

Substituting (17) and (18) into the equations of motion (8) and (10) and eliminating p^*, we obtain

$$\begin{aligned}
D^4\chi_1 = 2v_0\frac{\partial v_0}{\partial z}-r_1\frac{\partial^2}{\partial r_1\partial z_1}\left(\frac{v_0}{r_1}\right)&\left[4(\alpha_2'-\alpha_3')\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)-2\alpha_2'\left[r_1\frac{\partial^2}{\partial r_1^2}\left(\frac{v_0}{r_1}\right)-\frac{\partial^2 v_0}{\partial z_1^2}\right]\right]\\
&+2\frac{\partial v_0}{\partial z_1}\left[(\alpha_2'+\alpha_3')\frac{\partial^2}{\partial r_1^2}\left(\frac{v_0}{r_1}\right)+\frac{(4\alpha_2'+3\alpha_3')}{r_1}\frac{\partial^2 v_0}{\partial z_1^2}+\alpha_2' r_1\frac{\partial^3}{\partial r_1\partial z_1^2}\left(\frac{v_0}{r_1}\right)\right]\\
&+\frac{2}{r_1}\frac{\partial}{\partial r_1}\left(\frac{v_0}{r_1}\right)\left[3(\alpha_2'+\alpha_3')\frac{\partial v_0}{\partial r_1}-\alpha_2' r_1^3\frac{\partial^3}{\partial r_1^2\partial z_1}\left(\frac{v_0}{r_1}\right)\right],
\end{aligned}\tag{19}$$

where $$D^2 \equiv \frac{\partial^2}{\partial r_1^2}-\frac{1}{r_1}\frac{\partial}{\partial r_1}+\frac{\partial^2}{\partial z_1^2}.\tag{20}$$

Equations (14) and (19) *with a suitable choice of boundary conditions* can be used to predict the secondary flow patterns caused by the slow steady rotation of *any* solid of revolution in an elastico-viscous liquid.

(c) *Analytical approach based on 'infinite' plates*

The first step in attempting an explanation of the experimental results discussed in §2 would appear to be a consideration of the relatively simple *infinite*-plate problem, where the fluid is contained between two infinite parallel disks, one rotating about a vertical axis, the other being at rest. This approach is strongly suggested by experience with other geometries. For example, the rotating cone/stationary plate problem is conveniently solved by considering an infinite cone and an infinite plate (cf. Giesekus 1967; Walters & Waters 1968).

The main advantage in considering infinite disks is that a simple analytical solution to the problem can be found without difficulty, as we will now demonstrate.

The basic equations to order L are given by (14) and (19), where the boundary conditions are now

$$\left.\begin{aligned} \frac{\partial \chi_1}{\partial z_1} = \frac{\partial \chi_1}{\partial r_1} = 0, \quad v_0 = r_1 \quad \text{on} \quad z_1 = 1, \\ \frac{\partial \chi_1}{\partial z_1} = \frac{\partial \chi_1}{\partial r_1} = 0, \quad v_0 = 0 \quad \text{on} \quad z_1 = 0, \end{aligned}\right\} \tag{21}$$

and h is taken to be the gap between the plates. The solution of (14) subject to (21) is simply

$$v_0 = r_1 z_1 \tag{22}$$

and the rather complicated (19) reduces to (cf. Srivastava 1961; Bhatnagar 1963)

$$D^4\chi_1 = 2r_1^2 z_1. \tag{23}$$

The solution of (23) subject to (21) is

$$\chi_1 = r_1^2\left[\frac{z_1^5}{60} - \frac{z_1^3}{20} + \frac{z_1^2}{30}\right]. \tag{24}$$

With the simple geometry under consideration in the present section no difficulty is encountered in obtaining higher-order solutions. For example, the expression for v_1 can be shown to be

$$v_1 = r_1\left[-\frac{z_1^7}{315} + \frac{z_1^5}{100} - \frac{z_1^4}{180} - \frac{2z_1}{1575} + m\left(\frac{z_1^5}{30} - \frac{z_1^3}{10} + \frac{z_1^2}{15}\right)\right], \tag{25}$$

where $m = \alpha_2' + \alpha_3'$; and the equation for the second-order stream function χ_2 is

$$\begin{aligned} D^4\chi_2 = -r_1^2\Bigg[&\frac{8z_1}{1575} + \frac{z_1^2}{50} - \frac{3z_1^3}{100} - \frac{z_1^4}{90} - \frac{z_1^5}{100} + \frac{11z_1^7}{630} + \alpha_2'\left(-\frac{2}{25} + \frac{9z_1}{25} - \frac{4z_1^2}{15} + \frac{4z_1^5}{15}\right) \\ &+ \alpha_3'\left(-\frac{1}{50} + \frac{9z_1}{100} - \frac{3z_1^2}{5} + \frac{13z_1^3}{10} - \frac{7z_1^5}{10}\right) + 2(\alpha_5' + 2\beta_1')\left(\frac{2}{15} - \frac{3z_1}{5} + \frac{2z_1^3}{3}\right) \\ &+ 2(4\alpha_2' + \alpha_3')(\alpha_2' + \alpha_3')\left(\frac{2}{15} - \frac{3z_1}{5} + \frac{2z_1^3}{3}\right)\Bigg] - 4(\alpha_5' + \beta_1')r_1^4 z_1, \end{aligned} \tag{26}$$

with associated boundary conditions

$$\frac{\partial \chi_2}{\partial r_1} = \frac{\partial \chi_2}{\partial z_1} = 0 \quad \text{on} \quad z_1 = 0,\ z_1 = 1. \tag{27}$$

The solution to (26) subject to (27) is

$$\begin{aligned}\chi_2 = -r_1^2\Bigg[&\frac{332z_1^2}{4,536,000} - \frac{579z_1^3}{4,536,000} + \frac{z_1^5}{23,625} + \frac{z_1^6}{18,000} - \frac{z_1^7}{28,000} - \frac{z_1^8}{151,200}\\ &- \frac{z_1^9}{302,400} + \frac{z_1^{11}}{4,536,000} + \alpha_2'\left(\frac{23z_1^2}{23,625} + \frac{7z_1^3}{567,000} - \frac{z_1^4}{300} + \frac{3z_1^5}{1000} - \frac{z_1^6}{1350} + \frac{z_1^9}{11,340}\right)\\ &+ \alpha_3'\left(\frac{467z_1^2}{63,000} - \frac{323z_1^3}{63,000} - \frac{z_1^4}{1200} + \frac{3z_1^5}{4000} - \frac{z_1^6}{600} + \frac{13z_1^7}{8400} - \frac{z_1^9}{4320}\right)\\ &+ [(4\alpha_2' + \alpha_3')(\alpha_2' + \alpha_3') - (7\alpha_5' + 6\beta_1')]\left[-\frac{4z_1^2}{1575} - \frac{z_1^3}{6300} + \frac{z_1^4}{90} - \frac{z_1^5}{100} + \frac{z_1^7}{630}\right]\Bigg]\\ &- r_1^4\left[(\alpha_5' + \beta_1')\left(\frac{z_1^2}{15} - \frac{z_1^3}{10} + \frac{z_1^5}{30}\right)\right].\end{aligned} \tag{28}$$

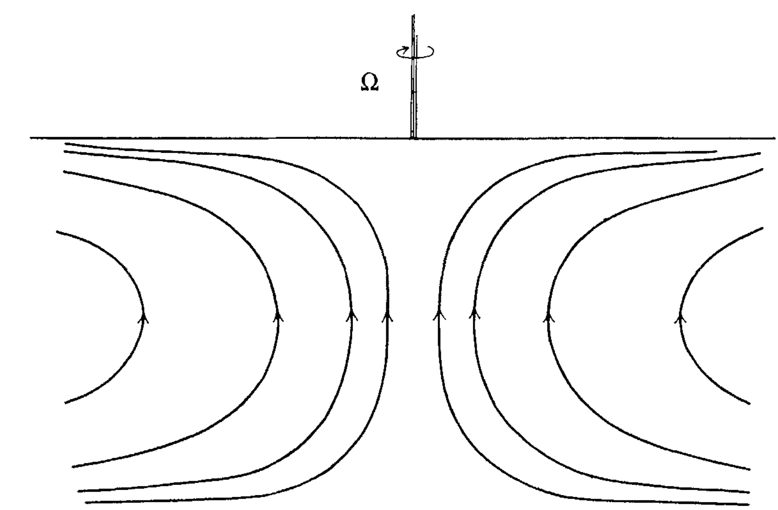

FIGURE 3. Streamline projections predicted on the basis of equation (24).

The solution for χ_1 and v_1 given by (24) and (25) are in agreement with the findings of Bhatnagar (1963), who considered the same problem for a Reiner–Rivlin fluid, which is a special case of the fluid considered here, obtained by writing $\alpha_2 = \alpha_5 = \beta_1 = \beta_2 = 0$ in (2). Bhatnagar did not consider higher-order terms in L and no further comparison is possible.

The streamline projections on any plane containing the axis of rotation are given by

$$\chi = \text{constant}.$$

Inspection of (24) reveals that to first order the streamline projections are independent of the elastic parameters. The form of the streamlines is given in figure 3. The streamlines are directed outwards at the rotating disk and inwards at the stationary disk and there is no 'elastic' effect of the type observed in the experiments.

Inspection of (28) indicates that the elastic parameters do affect the stream function to second order, but this is of academic interest only in the present context, since the experiments discussed in §2 are certainly within the conditions of the first-order theory (the value of L being approximately 0·005).

We conclude that the observed flow reversal throughout the flow field in the case of the elastico-viscous liquid must in some way be due to edge effects. That this is so is confirmed in the next section.

4. Numerical solution

In order to solve the rather complicated problem of the flow caused by the rotation of a disk of *finite* radius a in a bath of elastico-viscous liquid it is necessary to resort to a numerical procedure.

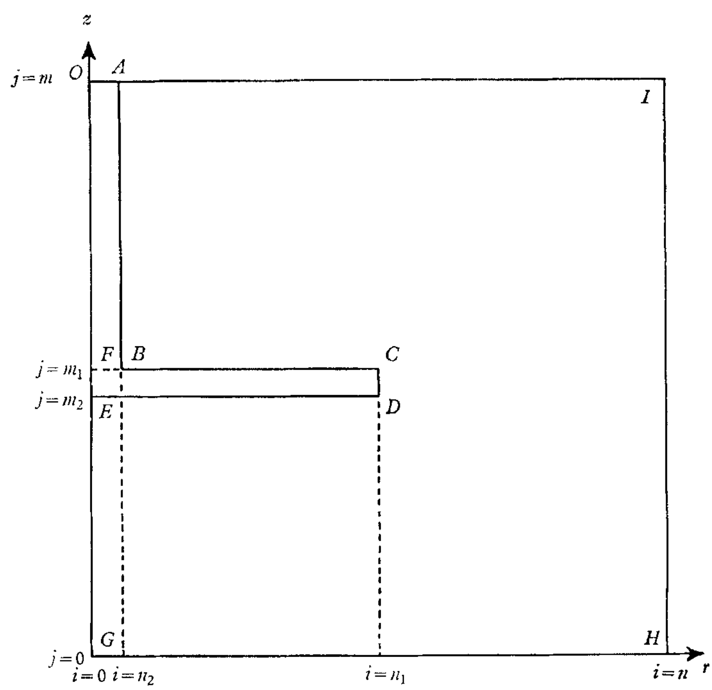

FIGURE 4

We consider the situation indicated in figure 4, which attempts to simulate the conditions of the experiments discussed in §2. $BCDE$ represents the rotating disk and $OABF$ the supporting shaft. The bath surface is given by GHI and AB is assumed large enough that the flow along AI may be represented by Couette flow. We have chosen this boundary condition in an attempt to simulate the conditions of the actual experiment. If we denote the interior of the region by R and the boundary by ∂R, it is convenient to divide ∂R into three parts ∂R_1, ∂R_2 and ∂R_3 which correspond to AI, $ABCDE$ and $EGHI$ respectively.

We now require a solution of (14) and (19) for v_0 and χ_1 subject to the boundary conditions

$$\left.\begin{aligned} v_0 &= C[r_1+D/r_1) && \text{on} \quad \partial R_1, \\ v_0 &= r_1 && \text{on} \quad \partial R_2, \\ v_0 &= 0 && \text{on} \quad \partial R_3, \end{aligned}\right\} \tag{29}$$

and

$$\chi_1 = \frac{\partial \chi_1}{\partial n} = 0 \quad \text{on} \quad \partial R, \tag{30}$$

where n denotes the direction normal to the boundary and

$$C = r_s^2/(r_s^2-r_b^2), \quad D = -r_b^2,$$

r_b and r_s being the non-dimensional radii of the bath and the rotating shaft respectively.

In this section it is more convenient to use the radius a of the disk as the typical length and accordingly all the non-dimensional parameters have to be modified. For example,

$$L = \left(\frac{\Omega a^2}{\nu}\right)^2, \quad \alpha_2' = \frac{\alpha_2}{\rho a^2}, \quad \beta_1' = \frac{\beta_1 \alpha_1}{\rho^2 a^4}, \quad \text{etc.}$$

(*a*) *Primary flow*

The primary flow equation (14) is independent of the elastic parameters and can therefore be solved once and for all. We first make the substitution

$$v_0 = r_1^{-\frac{1}{2}} g, \tag{31}$$

which reduces (14) to

$$E^2 g = 0, \tag{32}$$

where

$$E^2 \equiv \frac{\partial^2}{\partial r_1^2} - \frac{3}{4r_1^2} + \frac{\partial^2}{\partial z_1^2} \tag{33}$$

and the boundary conditions become

$$\left.\begin{aligned} g &= C(r_1^{\frac{3}{2}}+D/r_1^{\frac{1}{2}}) && \text{on} \quad \partial R_1, \\ g &= r_1^{\frac{3}{2}} && \text{on} \quad \partial R_2, \\ g &= 0 && \text{on} \quad \partial R_3. \end{aligned}\right\} \tag{34}$$

We now impose a mesh, $r_1 = i\Delta$, $z_1 = j\Delta$, over the region and denote $g(i\Delta, j\Delta)$ by g_{ij}. Equation (32) becomes, on using standard central difference representations,

$$\Delta^2 E_\Delta^2 g \equiv g_{i\,j-1} + g_{i-1\,j} - \left(4+\frac{3}{4i^2}\right) g_{ij} + g_{i+1\,j} + g_{i\,j+1} = 0, \tag{35}$$

where the operator E_Δ^2 denotes the discrete approximation to the operator E^2 given by (33). Using a Taylor series expansion, it can be shown that the conventional truncation error R_Δ of this equation is approximately

$$R_\Delta = \frac{\Delta^2}{12}\left(\frac{\partial^4 g}{\partial r_1^4} + \frac{\partial^4 g}{\partial z_1^4}\right). \tag{36}$$

However, it has been shown (Laasonen 1958) that for the Dirichlet problem in a region with re-entrant corners the truncation error is given by

$$R_\Delta = O(\Delta^{([2\pi/\alpha]-\epsilon)}), \tag{37}$$

where α is the greatest interior angle of the region and ϵ is any positive quantity. In the region R as defined above, (37) yields

$$R_\Delta = O(\Delta^{(\frac{4}{3}-\epsilon)}) \tag{38}$$

as the truncation error for (35).

The vector $\mathbf{g}_j$ is defined to be such that

$$\mathbf{g}_j = \{g_{lj},\quad g_{l+1\,j}, \ldots, g_{n-1\,j}\}^T,$$

where

$$\left.\begin{aligned} l &= 1 && \text{for} && 1 \leqslant j \leqslant m_2 - 1,\\ &= n_1 + 1 && \text{for} && m_2 \leqslant j \leqslant m_1,\\ &= n_2 + 1 && \text{for} && m_1 + 1 \leqslant j \leqslant m - 1, \end{aligned}\right\} \tag{39}$$

and n, n_1, n_2, m, m_1, m_2 are defined in figure 4.

On rearranging, (35) may be written in block matrix form

$$\begin{bmatrix} D_1 & E_1 & 0 & \ldots & \ldots \\ E_2 & D_2 & E_2 & \ldots & \ldots \\ \vdots & \vdots & \ddots & & \ldots \\ \ldots & \ldots & \ldots & E_{m-1} & D_{m-1} \end{bmatrix} \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_{m-1} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_{m-1} \end{bmatrix}, \tag{40}$$

where

$$D_j = \begin{bmatrix} 1 & -\dfrac{1}{C_l} & 0 & \ldots & 0 \\ -\dfrac{1}{C_{l+1}} & 1 & -\dfrac{1}{C_{l+1}} & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \ldots & \ldots & \ldots & -\dfrac{1}{C_{n-1}} & 1 \end{bmatrix},$$

$$E_j = \operatorname{diag}\left\{\frac{1}{C_l}, \quad \frac{1}{C_{l+1}}, \quad \ldots, \quad \frac{1}{C_{n-1}}\right\},$$

$$C_i = 4 + (3/4i^2)$$

and l is defined by (39).

The vector $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \quad \ldots, \quad \mathbf{b}_{m-1}\}^T$ is derived from the boundary conditions.

The linear set of algebraic equations (40) is now solved by the method of successive over-relaxational by lines (Forsythe & Wasaw 1960), in which the vector $\mathbf{g}_j$ at the kth iteration, denoted by $\mathbf{g}_j^{(k)}$, is derived from $\mathbf{g}_j^{(k-1)}$ by the equations

$$\left.\begin{aligned} E_j\mathbf{g}_{j-1}^{(k)} + D_j\mathbf{g}_j^{(k-\frac{1}{2})} + E_j\mathbf{g}_{j+1}^{(k-1)} &= \mathbf{b}_j,\\ \mathbf{g}_j^{(k)} = \omega\mathbf{g}_j^{(k-\frac{1}{2})} + (1-\omega)\mathbf{g}_j^{(k-1)}&, \end{aligned}\right\} \tag{41}$$

where ω lies in this range $1 < \omega < 2$ for over-relaxation.

The solution as given by (41) is convergent for $0 < \omega < 2$, since the matrix of coefficients may be made symmetric and irreducibly diagonally dominant (Varga 1962).

It is possible to test the numerical procedure described above by comparing the results with an analytic solution derived by Jeffery (1915) for the case of a finite disk rotating slowly in an infinite expanse of fluid. Figure 5 illustrates the agreement between the numerical results for v_0 (adapted to the case of an infinite expanse of fluid and no shaft) and Jeffery's analytic solution. The maximum deviation between the analytic and numerical curves is approximately one-fifth of a step length. This is within the error given in (37).

Having justified the numerical procedure, it is now possible to consider the more complicated geometry described in figure 4. Figure 6 contains the primary-flow velocity data for this situation. Also included are the corresponding curves for the *infinite* rotating plate case obtained from §3(*c*). In the next section it is shown that the difference between the finite-disk and infinite-disk curves can lead to gross changes in the secondary-flow characteristics.

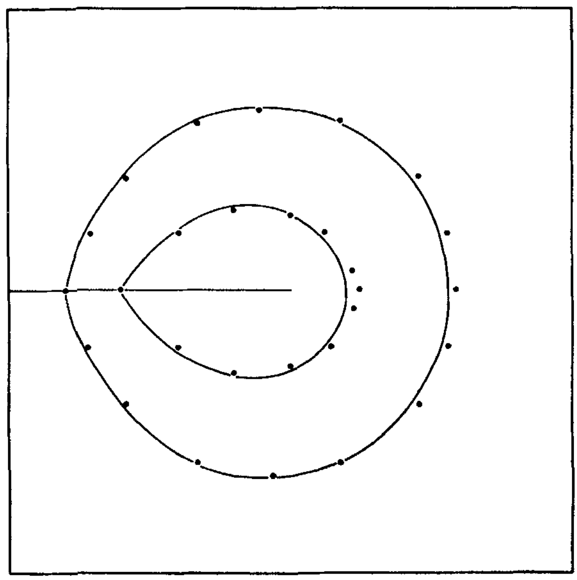

FIGURE 5. Curves of constant v_0 for finite disk based on numerical solution (●) compared with Jeffery's analytic solution (full line).

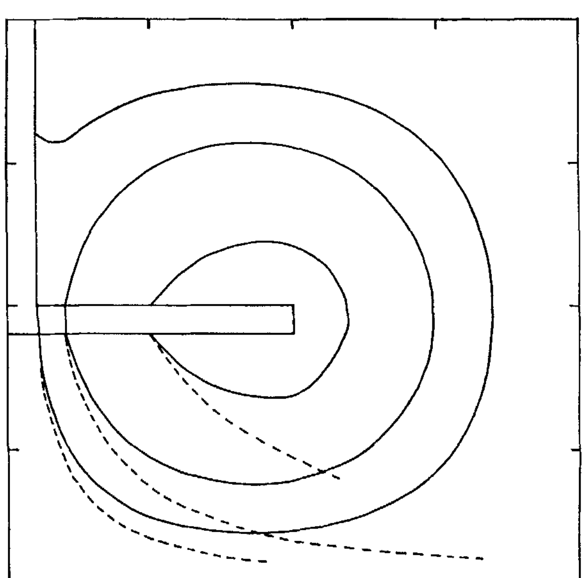

FIGURE 6. Curves of constant v_0 based on numerical solution (full line), compared with 'infinite-disk' prediction (broken line).

(b) *Secondary flow*

By means of the substitution

$$\chi_1 = r_1^{\frac{1}{2}}\psi, \tag{42}$$

(19) becomes

$$E^4\psi = r_1^{-\frac{1}{2}}F(v_0, r_1, z_1), \tag{43}$$

where $F(v_0, r_1, z_1)$ denotes the viscous and elastic forcing terms as given by the right-hand side of (19) and E^2 denotes the differential operator defined by (33).

Equation (43) is now written in the split operator form

$$E^2\psi = \phi, \tag{44}$$

$$E^2\phi = r_1^{-\frac{1}{2}}F(v_0, r_1, z_1) \tag{45}$$

and the associated boundary conditions are

$$\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad \partial R. \tag{46}$$

Equations (42), (44), (45) and (46) yield a complete solution for the stream function χ_1. Since no boundary conditions are given for ϕ, Taylor series approximations are used. As an illustration we consider the section of the boundary EG.

From (44) and (46) we have, in proceeding from the $(k-1)$th to the kth iteration,

$$\phi^{(k-\frac{1}{2})}(0, z_1) = \frac{\partial^2\psi^{(k-1)}}{\partial r_1^2}(0, z_1). \tag{47}$$

Using a Taylor series,

$$\phi^{(k-\frac{1}{2})}(0, z_1) = [8\psi^{(k-1)}(\Delta, z_1) - \psi^{(k-1)}(2\Delta, z_1)]/(2\Delta^2), \tag{48}$$

and a smoothing parameter s to avoid an unstable solution (see Pearson 1965), the boundary conditions for ϕ at the kth iteration become

$$\phi^{(k)}(0, z_1) = s\phi^{(k-1)}(0, z_1) + (1-s)\,\phi^{(k-\frac{1}{2})}(0, z_1). \tag{49}$$

The finite-difference forms of (44) and (45) are

$$E^2_\Delta\psi_{ij} = \Delta^2\phi_{ij}, \tag{50}$$

$$E^2_\Delta\phi_{ij} = \Delta^2F_{ij}(i\Delta)^{-\frac{1}{2}}, \tag{51}$$

where E^2_Δ is the discrete operator defined by (35). Equations (50) and (51) are now solved by a method similar to that described for the primary flow.

5. Numerical results

The method described in the last section has been used to obtain streamline projections for the situation described in figure 4. Although we shall include the whole of the region in the figures, our main concern is the region below the rotating disk.

Figure 7 illustrates the streamline projection in the viscous case, i.e. when $\alpha_2' = \alpha_3' = 0$. This figure compares favourably with the experimental results given in figure 1.

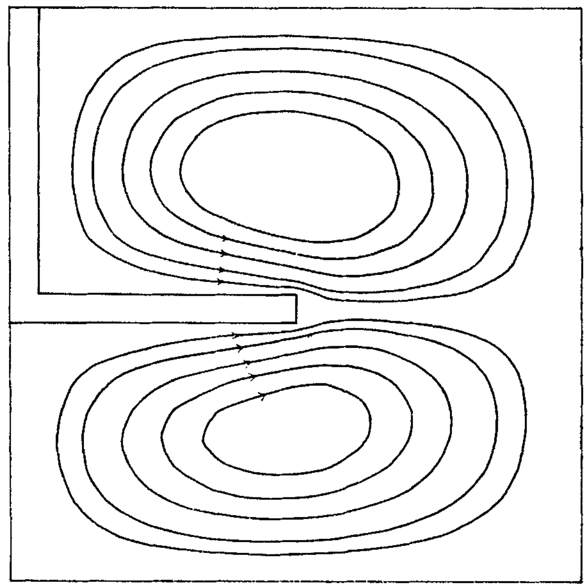

FIGURE 7. Predicted streamlines for a Newtonian liquid. ($\alpha_2' = \alpha_3' = 0$.)

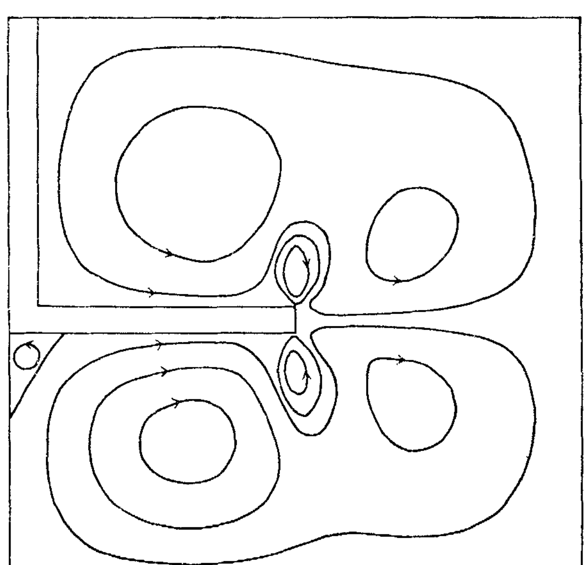

FIGURE 8. Predicted streamlines for a slightly elastic liquid with $\alpha_2' = -0{\cdot}02$, $\alpha_3' = 0{\cdot}04$.

Of more interest is the case of a slightly elastic liquid with $\alpha_2' = -0{\cdot}02$ and $\alpha_3' = 0{\cdot}04$ given in figure 8.† Here we note the appearance of a reversal vortex near the centre of the disk and a much stronger reversal vortex near the edge of the rotating disk. By increasing the elastic parameters it is found that these vortices grow in intensity (especially that near the edge of the disk) and finally join to give the situation shown in figure 9.‡ We conclude that the observed reversal in the secondary flow patterns is due to edge effects.

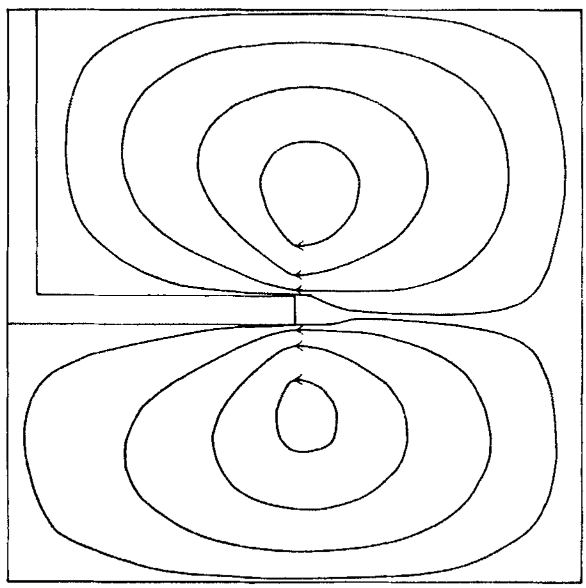

FIGURE 9. Predicted streamlines for an elastico-viscous liquid with $\alpha_2' = -1{\cdot}0$, $\alpha_3' = 2{\cdot}0$.

Having predicted the way that the reversal takes place, we thought it of some interest to demonstrate the intermediate state experimentally. Figure 10, plate 2, contains the streamline projections for a fairly viscous, slightly elastic, liquid (a 0·2 % solution of polyacrylamide in water 10 % + glycerol 90 %). The flow is very similar to that predicted in figure 8. Fluid is trapped near the centre of the disk and the reversal vortex near the edge of the disk is clearly discernible.

6. Conclusions

(i) The continuum theory, together with a numerical procedure, is able to predict rather complicated flow situations. (ii) Edge effects in the flow of elastic liquids can have quite a strong influence on flow characteristics and can, under

† It may be deduced on theoretical grounds that α_2' is negative and from experimental results that the most likely range of α_3' is $-2\alpha_2' \leqslant \alpha_3' < -2{\cdot}5\alpha_2'$. We have verified that varying α_3' between these limits does not significantly affect the general shape of the streamlines.

‡ Assigning the value -1 to α_2' is not unrealistic, since the polymer solution used in the second experiment had a limiting viscosity α_1 of approximately 50 poises, ρ was approximately 1·0 g/cm³, a was 4·75 cm and the major relaxation times could be estimated to be of the order of 0·5 sec. (For a liquid with just one relaxation time λ, $\alpha_2' = -\alpha_1\lambda/(\rho a^2)$.)

some conditions, affect the whole of a flow field. This may have applications in rheometry.

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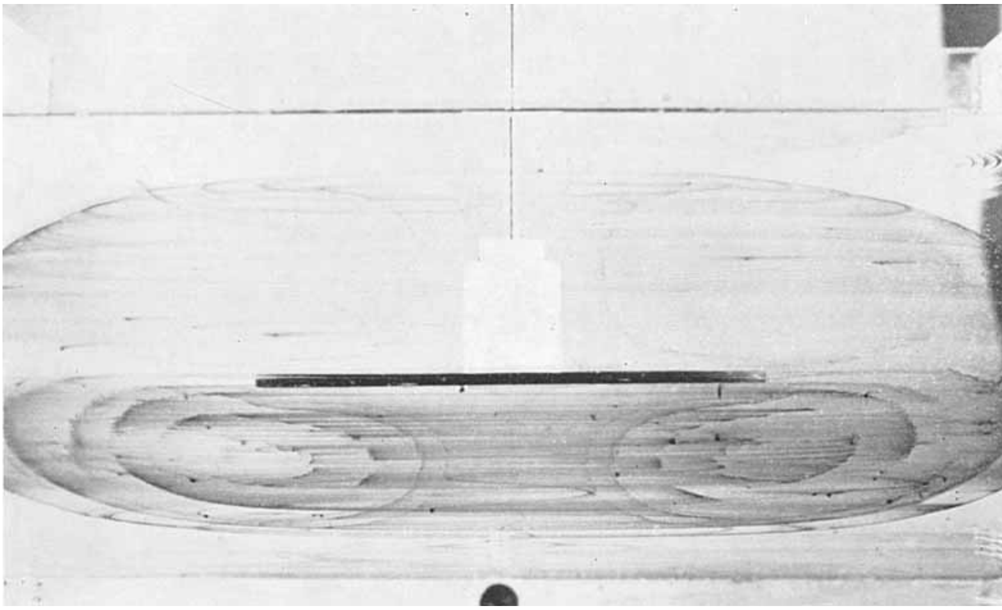

FIGURE 1. Streamlines for a Newtonian liquid (glycerol).

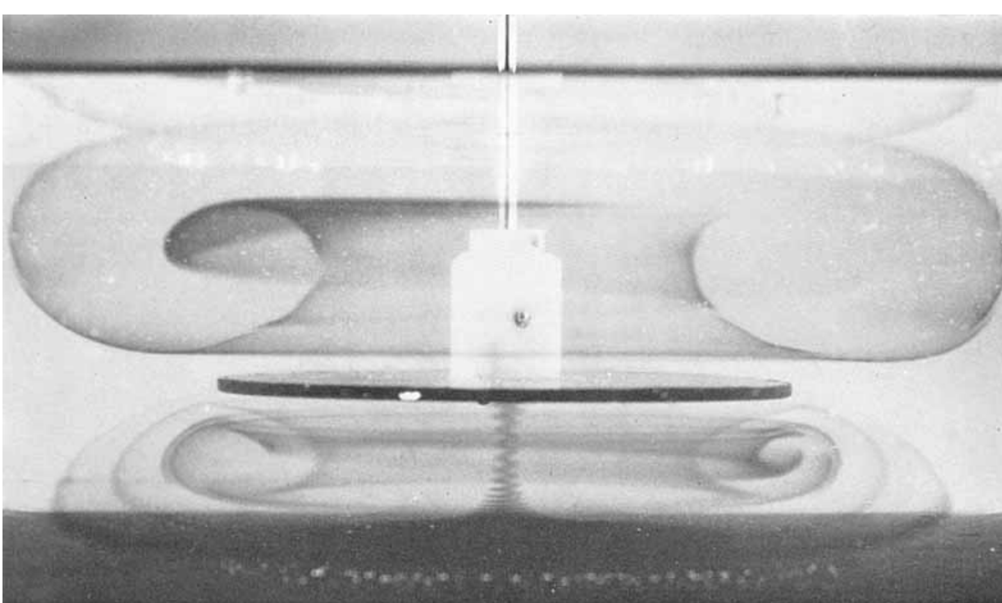

FIGURE 2. Streamlines for a 3·0 % aqueous solution of polyacrylamide.

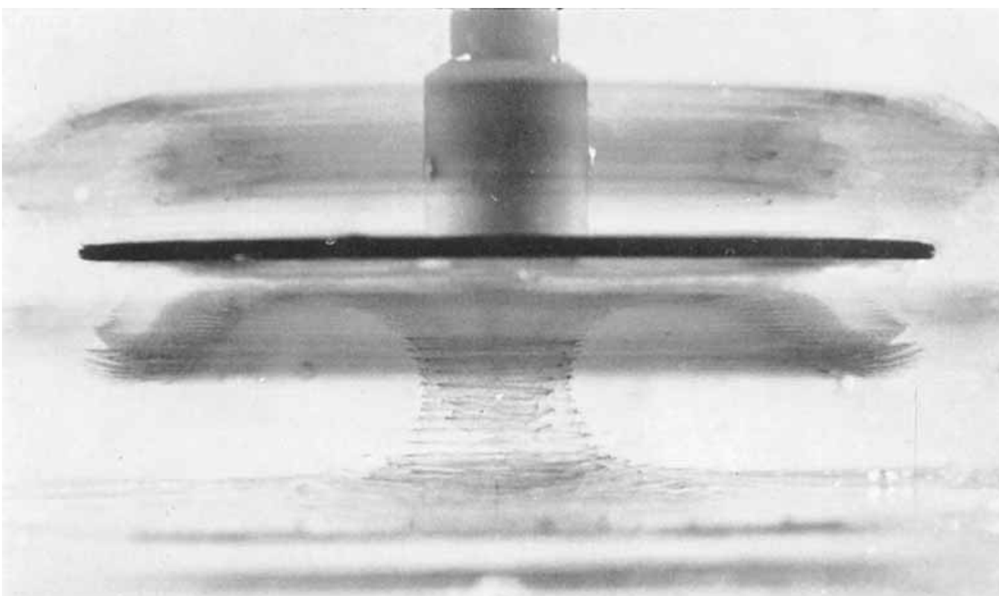

FIGURE 10. Observed streamlines for a slightly elastic liquid (0·2 % solution of polyacrylamide in water 10 % + glycerol 90 %).